1.5 Inverse Functions and Logarithms

You will be able to find inverses of one-to-one functions and will be able to analyze logarithmic functions algebraically, graphically, and numerically as inverses of exponential functions.

- One-to-one functions and the horizontal line test
- Finding inverse functions graphically and algebraically
- Base *a* logarithm functions
- Properties of logarithms
- Changing bases
- Using logarithms to solve exponential equations algebraically

One-to-One Functions

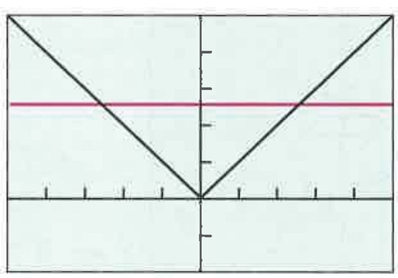
As you know, a function is a rule that assigns a single value in its range to each point in its domain. Some functions assign the same output to more than one input. For example, $f(x) = x^2$ assigns the output 4 to both 2 and -2. Other functions never output a given value more than once. For example, the cubes of different numbers are always different.

If each output value of a function is associated with exactly one input value, the function is *one-to-one*.

DEFINITION One-to-One Function

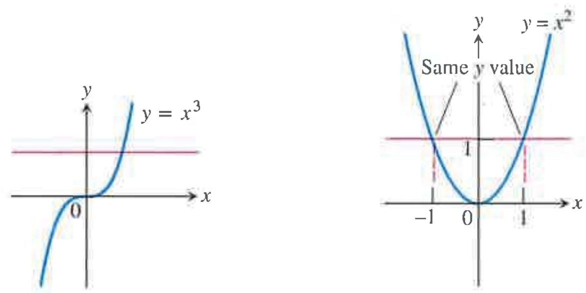
A function $f(x)$ is **one-to-one** on a domain D if $f(a) \neq f(b)$ whenever $a \neq b$.

The graph of a one-to-one function $y = f(x)$ can intersect any horizontal line at most once (the *horizontal line test*). If it intersects such a line more than once it assumes the same y value more than once, and is therefore not one-to-one (Figure 1.30).



[–5, 5] by [–2, 5]

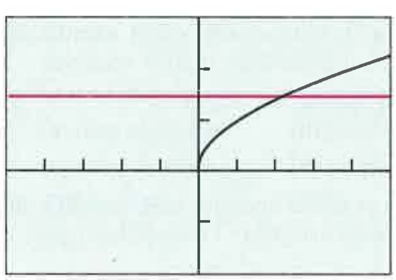
(a)



One-to-one: Graph meets each horizontal line once.

Not one-to-one: Graph meets some horizontal lines more than once.

Figure 1.30 Using the horizontal line test, we see that $y = x^3$ is one-to-one and $y = x^2$ is not.



[–5, 5] by [–2, 3]

(b)

Figure 1.31 (a) The graph of $f(x) = |x|$ and a horizontal line. (b) The graph of $g(x) = \sqrt{x}$ and a horizontal line. (Example 1)

EXAMPLE 1 Using the Horizontal Line Test

Determine whether the functions are one-to-one.

(a) $f(x) = |x|$ **(b)** $g(x) = \sqrt{x}$

SOLUTION

(a) As Figure 1.31a suggests, each horizontal line $y = c$, $c > 0$, intersects the graph of $f(x) = |x|$ twice. So f is not one-to-one.

(b) As Figure 1.31b suggests, each horizontal line intersects the graph of $g(x) = \sqrt{x}$ either once or not at all. The function g is one-to-one. *Now Try Exercise 1.*

Inverses

Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send outputs back to the inputs from which they came. The function defined by reversing a one-to-one function f is the **inverse of f.** The functions in Tables 1.2 and 1.3 are inverses of one another. The symbol for the inverse of f is f^{-1}, read "f inverse." The -1 in f^{-1} is not an exponent; $f^{-1}(x)$ does not mean $1/f(x)$.

TABLE 1.2
Rental Charge versus Time

Time x (hours)	Charge y (dollars)
1	5.00
2	7.50
3	10.00
4	12.50
5	15.00
6	17.50

TABLE 1.3
Time versus Rental Charge

Charge x (dollars)	Time y (hours)
5.00	1
7.50	2
10.00	3
12.50	4
15.00	5
17.50	6

As Tables 1.2 and 1.3 suggest, composing a function with its inverse in either order sends each output back to the input from which it came. In other words, the result of composing a function and its inverse in either order is the **identity function,** the function that assigns each number to itself. This gives a way to test whether two functions f and g are inverses of one another. Compute $f \circ g$ and $g \circ f$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then f and g are inverses of one another; otherwise they are not. The functions $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of one another because $(x^3)^{1/3} = x$ and $(x^{1/3})^3 = x$ for every number x.

EXPLORATION 1 Testing for Inverses Graphically

For each of the function pairs below,

(a) Graph f and g together in a square window.

(b) Graph $f \circ g$. **(c)** Graph $g \circ f$.

What can you conclude from the graphs?

1. $f(x) = x^3, \quad g(x) = x^{1/3}$

2. $f(x) = x, \quad g(x) = 1/x$

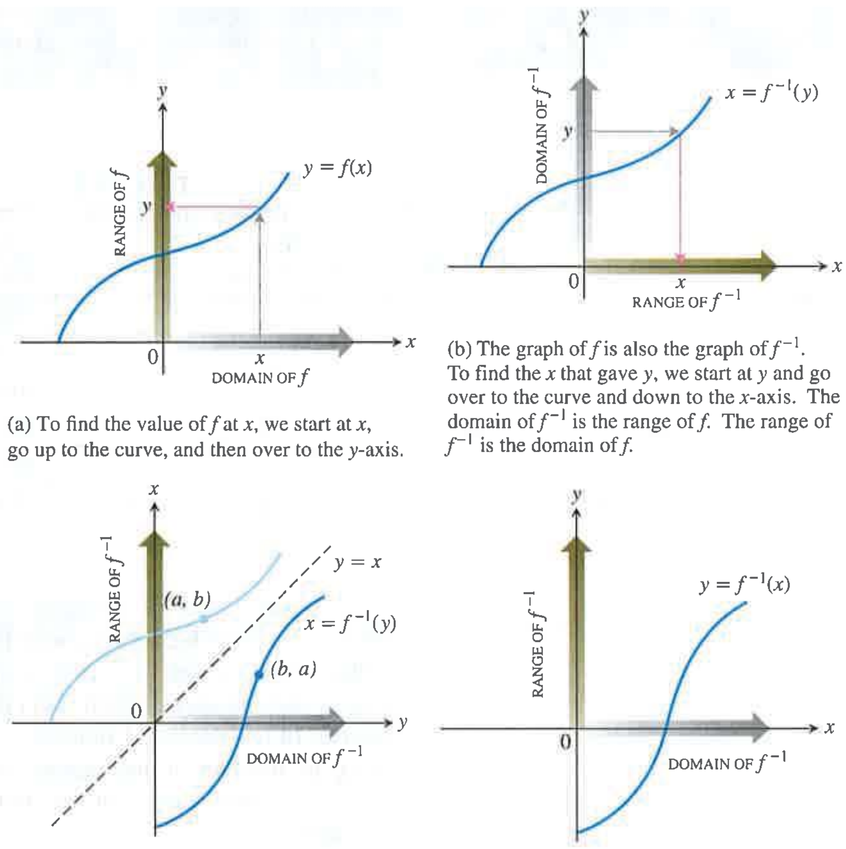
3. $f(x) = 3x, \quad g(x) = x/3$

4. $f(x) = e^x, \quad g(x) = \ln x$

Finding Inverses

How do we find the graph of the inverse of a function? Suppose, for example, that the function is the one pictured in Figure 1.32a. To read the graph, we start at the point x on the x-axis, go up to the graph, and then move over to the y-axis to read the value of y. If we start with y and want to find the x from which it came, we reverse the process (Figure 1.32b).

The graph of f is already the graph of f^{-1}, although the latter graph is not drawn in the usual way with the domain axis horizontal and the range axis vertical. For f^{-1}, the input-output pairs are reversed. To display the graph of f^{-1} in the usual way, we have to reverse the pairs by reflecting the graph across the 45° line $y = x$ (Figure 1.32c) and interchanging the letters x and y (Figure 1.32d). This puts the independent variable, now called x, on the horizontal axis and the dependent variable, now called y, on the vertical axis.



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f is also the graph of f^{-1}. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the usual way, we reflect the system across the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

Figure 1.32 The graph of $y = f^{-1}(x)$.

The fact that the graphs of f and f^{-1} are reflections of each other across the line $y = x$ is to be expected because the input-output pairs (a, b) of f have been reversed to produce the input-output pairs (b, a) of f^{-1}.

The pictures in Figure 1.32 tell us how to express f^{-1} as a function of x algebraically.

Writing f^{-1} as a Function of x

1. Solve the equation $y = f(x)$ for x in terms of y.
2. Interchange x and y. The resulting formula will be $y = f^{-1}(x)$.

EXAMPLE 2 Finding the Inverse Function

Show that the function $y = f(x) = -2x + 4$ is one-to-one and find its inverse function.

SOLUTION

Every horizontal line intersects the graph of f exactly once, so f is one-to-one and has an inverse.

Step 1:

Solve for x in terms of y: $y = -2x + 4$

$$x = -\frac{1}{2}y + 2$$

continued

Step 2:

Interchange x and y: $y = -\frac{1}{2}x + 2$

The inverse of the function $f(x) = -2x + 4$ is the function $f^{-1}(x) = -(1/2)x + 2$. We can verify that both composites are the identity function.

$$f^{-1}(f(x)) = -\frac{1}{2}(-2x + 4) + 2 = x - 2 + 2 = x$$

$$f(f^{-1}(x)) = -2\left(-\frac{1}{2}x + 2\right) + 4 = x - 4 + 4 = x$$

Now Try Exercise 13.

Graphing $y = f(x)$ and $y = f^{-1}(x)$ Parametrically

We can graph any function $y = f(x)$ as

$$x_1 = t, \quad y_1 = f(t).$$

Interchanging t and $f(t)$ produces parametric equations for the inverse:

$$x_2 = f(t), \quad y_2 = t$$

We can use parametric graphing to graph the inverse of a function without finding an explicit rule for the inverse, as illustrated in Example 3.

EXAMPLE 3 Graphing the Inverse Parametrically

(a) Graph the one-to-one function $f(x) = x^2$, $x \geq 0$, together with its inverse and the line $y = x$, $x \geq 0$.

(b) Express the inverse of f as a function of x.

SOLUTION

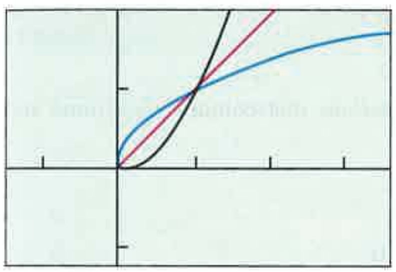
(a) We can graph the three functions parametrically as follows:

Graph of f: $x_1 = t, \quad y_1 = t^2, \quad t \geq 0$

Graph of f^{-1}: $x_2 = t^2, \quad y_2 = t$

Graph of $y = x$: $x_3 = t, \quad y_3 = t$

Figure 1.33 shows the three graphs.



[–1.5, 3.5] by [–1, 2]

Figure 1.33 The graphs of f and f^{-1} are reflections of each other across the line $y = x$. (Example 3)

(b) Next we find a formula for $f^{-1}(x)$.

Step 1:

Solve for x in terms of y.

$$y = x^2$$
$$\sqrt{y} = \sqrt{x^2}$$
$$\sqrt{y} = x \qquad \text{Because } x \geq 0$$

Step 2:

Interchange x and y.

$$\sqrt{x} = y$$

Thus, $f^{-1}(x) = \sqrt{x}$.

Now Try Exercise 27.

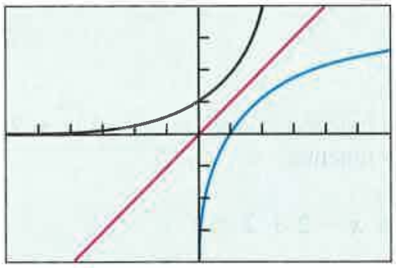
Logarithmic Functions

If a is any positive real number other than 1, the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *base a logarithm function.*

DEFINITION Base *a* Logarithm Function

The **base *a* logarithm function** $y = \log_a x$ is the inverse of the base a exponential function $y = a^x (a > 0, a \neq 1)$.

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x. The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x.



[–6, 6] by [–4, 4]

Figure 1.34 The graphs of $y = 2^x$ ($x_1 = t$, $y_1 = 2^t$), its inverse $y = \log_2 x$ ($x_2 = 2^t$, $y_2 = t$), and $y = x$ ($x_3 = t$, $y_3 = t$).

Because we have no technique for solving for x in terms of y in the equation $y = a^x$, we do not have an explicit formula for the logarithm function as a function of x. However, the graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$, or by using parametric graphing (Figure 1.34).

Logarithms with base e and base 10 are so important in applications that calculators have special keys for them. They also have their own special notation and names:

$$\log_e x = \ln x,$$
$$\log_{10} x = \log x$$

The function $y = \ln x$ is called the **natural logarithm function** and $y = \log x$ is often called the **common logarithm function.**

Properties of Logarithms

Because a^x and $\log_a x$ are inverses of each other, composing them in either order gives the identity function. This gives two useful properties.

Inverse Properties for a^x and $\log_a x$

1. Base $a > 0$ and $a \neq 1$: $a^{\log_a x} = x$ for $x > 0$; $\log_a a^x = x$ for all x
2. Base e: $e^{\ln x} = x$ for $x > 0$; $\ln e^x = x$ for all x

These properties help us with the solution of equations that contain logarithms and exponential functions.

EXAMPLE 4 Using the Inverse Properties

Solve for x: **(a)** $\ln x = 3t + 5$ **(b)** $e^{2x} = 10$

SOLUTION

(a) $\ln x = 3t + 5$

$e^{\ln x} = e^{3t+5}$ Exponentiate both sides.

$x = e^{3t+5}$ Inverse Property

(b) $e^{2x} = 10$

$\ln e^{2x} = \ln 10$ Take logarithms of both sides.

$2x = \ln 10$ Inverse Property

$x = \frac{1}{2} \ln 10 \approx 1.15$

Now Try Exercises 33 and 37.

The logarithm function has the following useful arithmetic properties.

Properties of Logarithms

For any real numbers $x > 0$ and $y > 0$,

1. *Product Rule:* $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:* $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Power Rule:* $\log_a x^y = y \log_a x$

EXPLORATION 2 Supporting the Product Rule

Let $y_1 = \ln(ax)$, $y_2 = \ln x$, and $y_3 = y_1 - y_2$.

1. Graph y_1 and y_2 for $a = 2, 3, 4,$ and 5. How do the graphs of y_1 and y_2 appear to be related?
2. Support your finding by graphing y_3.
3. Confirm your finding algebraically.

The following formula allows us to evaluate $\log_a x$ for any base $a > 0$, $a \neq 1$, and to obtain its graph using the natural logarithm function on our grapher.

Change of Base Formula

$$\log_a x = \frac{\ln x}{\ln a}$$

Changing the Base for Changing the Base

The log button on your calculator works just as well as the ln button in the change of base formula:

$$\log_a x = \frac{\log x}{\log a}$$

EXAMPLE 5 Graphing a Base *a* Logarithm Function

Graph $f(x) = \log_2 x$.

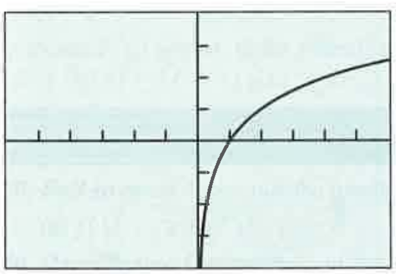
SOLUTION

We use the change of base formula to rewrite $f(x)$.

$$f(x) = \log_2 x = \frac{\ln x}{\ln 2}$$

Figure 1.35 gives the graph of f. ***Now Try Exercise 41.***

$y = \dfrac{\ln x}{\ln 2}$



[–6, 6] by [–4, 4]

Figure 1.35 The graph of $f(x) = \log_2 x$ using $f(x) = (\ln x)/(\ln 2)$. (Example 5)

Applications

In Section 1.3 we used graphical methods to solve exponential growth and decay problems. Now we can use the properties of logarithms to solve the same problems algebraically.

EXAMPLE 6 Finding Time

Sarah invests \$1000 in an account that earns 5.25% interest compounded annually. How long will it take the account to reach \$2500?

SOLUTION

The amount in the account at any time t in years is $1000(1.0525)^t$, so we need to solve the equation

$$1000(1.0525)^t = 2500.$$

continued

$$(1.0525)^t = 2.5 \qquad \text{Divide by 1000.}$$

$$\ln(1.0525)^t = \ln 2.5 \qquad \text{Take logarithms of both sides.}$$

$$t \ln 1.0525 = \ln 2.5 \qquad \text{Power Rule}$$

$$t = \frac{\ln 2.5}{\ln 1.0525} \approx 17.9$$

The amount in Sarah's account will be \$2500 in about 17.9 years, or about 17 years and 11 months.

Now Try Exercise 47.

Quick Review 1.5 *(For help, go to Sections 1.2, 1.3, and 1.4.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, let $f(x) = \sqrt[3]{x - 1}$, $g(x) = x^2 + 1$, and evaluate the expression.

1. $(f \circ g)(1)$

2. $(g \circ f)(-7)$

3. $(f \circ g)(x)$

4. $(g \circ f)(x)$

In Exercises 5 and 6, choose parametric equations and a parameter interval to represent the function on the interval specified.

5. $y = \dfrac{1}{x - 1}, \quad x \geq 2$

6. $y = x, \quad x < -3$

In Exercises 7–10, find the points of intersection of the two curves. Round your answers to 2 decimal places.

7. $y = 2x - 3, \quad y = 5$

8. $y = -3x + 5, \quad y = -3$

9. **(a)** $y = 2^x, \quad y = 3$

(b) $y = 2^x, \quad y = -1$

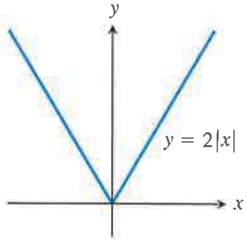
10. **(a)** $y = e^{-x}, \quad y = 4$

(b) $y = e^{-x}, \quad y = -1$

Section 1.5 Exercises

In Exercises 1–6, determine whether the function is one-to-one.

1.



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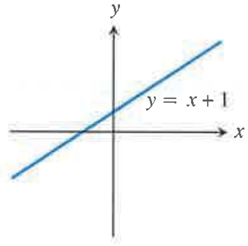

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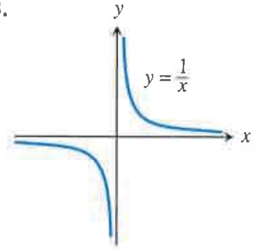

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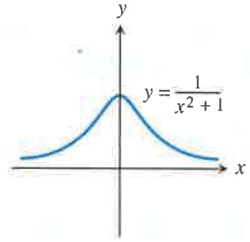
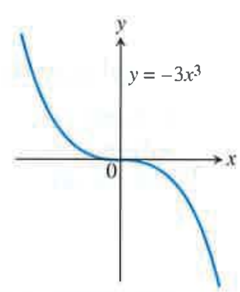
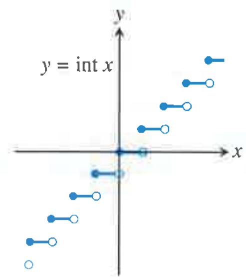


5.



6.



In Exercises 7–12, determine whether the function has an inverse function.

7. $y = \dfrac{3}{x - 2} - 1$

8. $y = x^2 + 5x$

9. $y = x^3 - 4x + 6$

10. $y = x^3 + x$

11. $y = \ln x^2$

12. $y = 2^{3-x}$

In Exercises 13–24, find f^{-1} and verify that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x.$$

13. $f(x) = 2x + 3$

14. $f(x) = 5 - 4x$

15. $f(x) = x^3 - 1$

16. $f(x) = x^2 + 1, \quad x \geq 0$

17. $f(x) = x^2, \quad x \leq 0$

18. $f(x) = x^{2/3}, \quad x \geq 0$

19. $f(x) = -(x - 2)^2, \quad x \leq 2$

20. $f(x) = x^2 + 2x + 1, \quad x \geq -1$

21. $f(x) = \dfrac{1}{x^2}, \quad x > 0$

22. $f(x) = \dfrac{1}{x^3}$

23. $f(x) = \dfrac{2x + 1}{x + 3}$

24. $f(x) = \dfrac{x + 3}{x - 2}$

In Exercises 25–32, use parametric graphing to graph f, f^{-1}, and $y = x$.

25. $f(x) = e^x$

26. $f(x) = 3^x$

27. $f(x) = 2^{-x}$

28. $f(x) = 3^{-x}$

29. $f(x) = \ln x$

30. $f(x) = \log x$

31. $f(x) = \sin^{-1} x$

32. $f(x) = \tan^{-1} x$

In Exercises 33–36, solve the equation algebraically. You can check your solution graphically.

33. $(1.045)^t = 2$

34. $e^{0.05t} = 3$

35. $e^x + e^{-x} = 3$

36. $2^x + 2^{-x} = 5$

In Exercises 37 and 38, solve for y.

37. $\ln y = 2t + 4$

38. $\ln (y - 1) - \ln 2 = x + \ln x$

In Exercises 39–42, draw the graph and determine the domain and range of the function.

39. $y = 2 \ln (3 - x) - 4$

40. $y = -3 \log (x + 2) + 1$

41. $y = \log_2 (x + 1)$

42. $y = \log_3 (x - 4)$

In Exercises 43 and 44, find a formula for f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

43. $f(x) = \dfrac{100}{1 + 2^{-x}}$

44. $f(x) = \dfrac{50}{1 + 1.1^{-x}}$

45. ***Self-inverse*** Prove that the function f is its own inverse.

(a) $f(x) = \sqrt{1 - x^2}, \quad x \geq 0$ **(b)** $f(x) = 1/x$

46. ***Radioactive Decay*** The half-life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

(a) Express the amount of substance remaining as a function of time t.

(b) When will there be 1 gram remaining?

47. ***Doubling Your Money*** Determine how much time is required for a \$500 investment to double in value if interest is earned at the rate of 4.75% compounded annually.

48. ***Population Growth*** The population of Glenbrook is 375,000 and is increasing at the rate of 2.25% per year. Predict when the population will be 1 million.

49. ***Guess the Curve*** A curve is defined parametrically as the set of points $(\sqrt{2 - t}, \sqrt{2 + t})$ for $-2 \leq t \leq 2$. Answer parts (a) through (d) before using your grapher.

(a) Explain why this parametrization cannot be used for other values of t.

(b) If a point is on this curve, what is its distance from the origin?

(c) Find the endpoints of the curve (determined by $t = -2$ and $t = 2$, respectively).

(d) Explain why all other points on this curve must lie in the first quadrant.

(e) Based on what you know from (a) through (d), give a complete geometric description of the curve. Then verify your answer with your grapher.

50. ***Logarithmic Equations*** For an algebraic challenge, solve these equations *without a calculator* by using the Laws of Logarithms.

(a) $4 \ln \sqrt{e^x} = 26$

(b) $x - \log (100) = \ln (e^3)$

(c) $\log_x (7x - 10) = 2$

(d) $2 \log_3 x - \log_3 (x - 2) = 2$

51. **Group Activity** ***Inverse Functions*** Let $y = f(x) = mx + b$, $m \neq 0$.

(a) **Writing to Learn** Give a convincing argument that f is a one-to-one function.

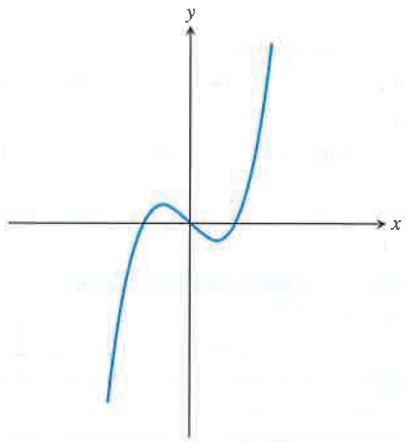
(b) Find a formula for the inverse of f. How are the slopes of f and f^{-1} related?

(c) If the graphs of two functions are parallel lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

(d) If the graphs of two functions are perpendicular lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

Standardized Test Questions

52. True or False The function displayed in the graph below is one-to-one. Justify your answer.



53. True or False If $(f \circ g)(x) = x$, then g is the inverse function of f. Justify your answer.

In Exercises 54 and 55, use the function $f(x) = 3 - \ln(x + 2)$.

54. Multiple Choice Which of the following is the domain of f?

(A) $x \neq -2$ (B) $(-\infty, \infty)$ (C) $(-2, \infty)$

(D) $[-1.9, \infty)$ (E) $(0, \infty)$

55. Multiple Choice Which of the following is the range of f?

(A) $(-\infty, \infty)$ (B) $(-\infty, 0)$ (C) $(-2, \infty)$

(D) $(0, \infty)$ (E) $(0, 5.3)$

56. Multiple Choice Which of the following is the inverse of $f(x) = 3x - 2$?

(A) $g(x) = \dfrac{1}{3x - 2}$ (B) $g(x) = x$ (C) $g(x) = 3x - 2$

(D) $g(x) = \dfrac{x - 2}{3}$ (E) $g(x) = \dfrac{x + 2}{3}$

57. Multiple Choice Which of the following is a solution of the equation $2 - 3^{-x} = -1$?

(A) $x = -2$ (B) $x = -1$ (C) $x = 0$

(D) $x = 1$ (E) There are no solutions.

Exploration

58. *Supporting the Quotient Rule* Let $y_1 = \ln(x/a)$, $y_2 = \ln x$, $y_3 = y_2 - y_1$, and $y_4 = e^{y_3}$.

(a) Graph y_1 and y_2 for $a = 2, 3, 4$, and 5. How are the graphs of y_1 and y_2 related?

(b) Graph y_3 for $a = 2, 3, 4$, and 5. Describe the graphs.

(c) Graph y_4 for $a = 2, 3, 4$, and 5. Compare the graphs to the graph of $y = a$.

(d) Use $e^{y_3} = e^{y_2 - y_1} = a$ to solve for y_1.

Extending the Ideas

59. *One-to-One Functions* If f is a one-to-one function, prove that $g(x) = -f(x)$ is also one-to-one.

60. *One-to-One Functions* If f is a one-to-one function and $f(x)$ is never zero, prove that $g(x) = 1/f(x)$ is also one-to-one.

61. *Domain and Range* Suppose that $a \neq 0$, $b \neq 1$, and $b > 0$. Determine the domain and range of the function.

(a) $y = a(b^{c-x}) + d$

(b) $y = a \log_b (x - c) + d$

62. Group Activity *Inverse Functions*

Let $f(x) = \dfrac{ax + b}{cx + d}$, $\quad c \neq 0, \quad ad - bc \neq 0.$

(a) Writing to Learn Give a convincing argument that f is one-to-one.

(b) Find a formula for the inverse of f.

(c) Find the horizontal and vertical asymptotes of f.

(d) Find the horizontal and vertical asymptotes of f^{-1}. How are they related to those of f?