1.4 Parametric Equations

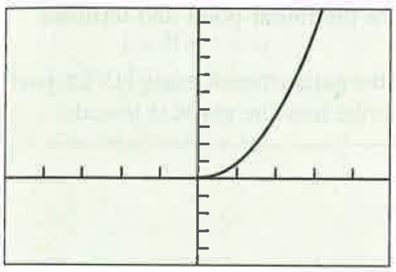
You will be able to analyze functions and relations defined parametrically and know how to determine their graphs; in particular, you will be able to analyze inverse relations algebraically and graphically by switching parametrizations of x and y.

- Parametrically defined curves
- Parametrizations of simple curves (lines and segments, circles, ellipses)
- The witch of Agnesi

Relations

A **relation** is a set of ordered pairs (x, y) of real numbers. The **graph of a relation** is the set of points in the plane that correspond to the ordered pairs of the relation. If x and y are *functions* of a third variable t, called a *parameter*, then we can use the *parametric mode* of a grapher to obtain a graph of the relation.

$x = \sqrt{t}, y = t$



[−5, 5] by [−5, 10]

Figure 1.25 You must choose a *smallest* and *largest* value for t in parametric mode. Here we used 0 and 10, respectively. (Example 1)

EXAMPLE 1 Graphing Half a Parabola

Describe the graph of the relation determined by

$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Set $x_1 = \sqrt{t}$, $y_1 = t$, and use the parametric mode of the grapher to draw the graph in Figure 1.25. The graph appears to be the right half of the parabola $y = x^2$. Notice that there is no information about t on the graph itself. The curve appears to be traced to the upper right with starting point $(0, 0)$.

Both x and y will be greater than or equal to zero because $t \geq 0$. Eliminating t we find that for every value of t,

$$y = t = (\sqrt{t})^2 = x^2.$$

Thus, the relation is the function $y = x^2, x \geq 0$. *Now Try Exercise 5.*

DEFINITIONS Parametric Curve, Parametric Equations

If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval of t values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve.** The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve and its domain I is the **parameter interval.** If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point of the curve** and the point $(f(b), g(b))$ is the **terminal point of the curve.** When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval constitute a **parametrization of the curve.**

In Example 1, the parameter interval is $[0, \infty)$, so $(0, 0)$ is the initial point and there is no terminal point.

A grapher can draw a parametrized curve only over a closed interval, so the portion it draws has endpoints even when the curve being graphed does not. Keep this in mind when you graph.

Circles

In applications, t often denotes time, an angle, or the distance a particle has traveled along its path from its starting point. In fact, parametric graphing can be used to simulate the motion of the particle.

EXPLORATION 1 Parametrizing Circles

Let $x = a \cos t$ and $y = a \sin t$.

1. Let $a = 1, 2,$ or 3 and graph the parametric equations in a *square viewing window* using the parameter interval $[0, 2\pi]$. How does changing a affect this graph?
2. Let $a = 2$ and graph the parametric equations using the following parameter intervals: $[0, \pi/2]$, $[0, \pi]$, $[0, 3\pi/2]$, $[2\pi, 4\pi]$, and $[0, 4\pi]$. Describe the role of the length of the parameter interval.
3. Let $a = 3$ and graph the parametric equations using the intervals $[\pi/2, 3\pi/2]$, $[\pi, 2\pi]$, $[3\pi/2, 3\pi]$, and $[\pi, 5\pi]$. What are the initial point and terminal point in each case?
4. Graph $x = 2\cos(-t)$ and $y = 2\sin(-t)$ using the parameter intervals $[0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. In each case, describe how the graph is traced.

For $x = a \cos t$ and $y = a \sin t$, we have

$$x^2 + y^2 = a^2\cos^2 t + a^2 \sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2(1) = a^2,$$

using the identity $\cos^2 t + \sin^2 t = 1$. Thus, the curves in Exploration 1 were either circles or portions of circles, each with center at the origin.

EXAMPLE 2 Graphing a Circle

Describe the graph of the relation determined by

$$x = 2\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi.$$

Find the initial and terminal points, if any, and indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Figure 1.26 shows that the graph appears to be a circle with radius 2. By watching the graph develop we can see that the curve is traced exactly once counterclockwise. The initial point at $t = 0$ is $(2, 0)$, and the terminal point at $t = 2\pi$ is also $(2, 0)$.

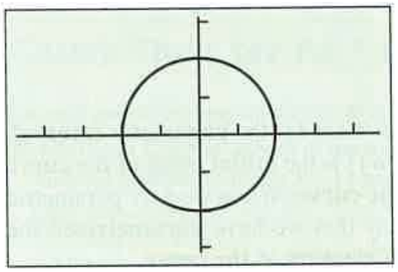
Next we eliminate the variable t.

$$\begin{aligned} x^2 + y^2 &= 4\cos^2 t + 4\sin^2 t \\ &= 4(\cos^2 t + \sin^2 t) \\ &= 4 \qquad \text{Because } \cos^2 t + \sin^2 t = 1 \end{aligned}$$

The parametrized curve is a circle centered at the origin of radius 2.

Now Try Exercise 9.

$x = 2\cos t, y = 2\sin t$



[–4.7, 4.7] by [–3.1, 3.1]

Figure 1.26 A graph of the parametric curve $x = 2\cos t$, $y = 2\sin t$, with Tmin $= 0$, Tmax $= 2\pi$, and Tstep $= \pi/24 \approx 0.131$. (Example 2)

Ellipses

Parametrizations of ellipses are similar to parametrizations of circles. Recall that the standard form of an ellipse centered at $(0, 0)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

EXAMPLE 3 Graphing an Ellipse

Graph the parametric curve $x = 3 \cos t$, $y = 4 \sin t$, $0 \le t \le 2\pi$.

Find a Cartesian equation for a curve that contains the parametric curve. What portion of the graph of the Cartesian equation is traced by the parametric curve? Indicate the direction in which the curve is traced and the initial and terminal points, if any.

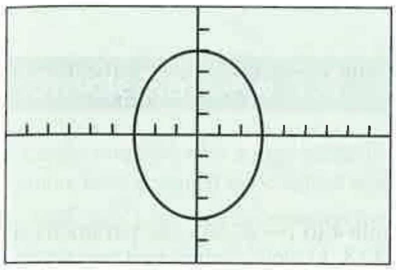
SOLUTION

Figure 1.27 suggests that the curve is an ellipse. The Cartesian equation is

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \cos^2 t + \sin^2 t = 1,$$

so the parametrized curve lies along an ellipse with major axis endpoints $(0, \pm 4)$ and minor axis endpoints $(\pm 3, 0)$. As t increases from 0 to 2π, the point $(x, y) = (3 \cos t, 4 \sin t)$ starts at $(3, 0)$ and traces the entire ellipse once counterclockwise. Thus, $(3, 0)$ is both the initial point and the terminal point. *Now Try Exercise 13.*

$x = 3 \cos t$, $y = 4 \sin t$



[–9, 9] by [–6, 6]

Figure 1.27 A graph of the parametric equations $x = 3 \cos t$, $y = 4 \sin t$ for $0 \le t \le 2\pi$. (Example 3)

EXPLORATION 2 Parametrizing Ellipses

Let $x = a \cos t$ and $y = b \sin t$.

1. Let $a = 2$ and $b = 3$. Then graph using the parameter interval $[0, 2\pi]$. Repeat, changing b to 4, 5, and 6.
2. Let $a = 3$ and $b = 4$. Then graph using the parameter interval $[0, 2\pi]$. Repeat, changing a to 5, 6, and 7.
3. Based on parts 1 and 2, how do you identify the axis that contains the major axis of the ellipse? the minor axis?
4. Let $a = 4$ and $b = 3$. Then graph using the parameter intervals $[0, \pi/2]$, $[0, \pi]$, $[0, 3\pi/2]$, and $[0, 4\pi]$. Describe the role of the length of the parameter interval.
5. Graph $x = 5 \cos(-t)$ and $y = 2 \sin(-t)$ using the parameter intervals $(0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. Describe how the graph is traced. What are the initial point and terminal point in each case?

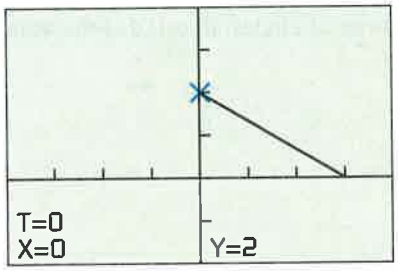
For $x = a \cos t$ and $y = b \sin t$, we have $(x/a)^2 + (y/b)^2 = \cos^2 t + \sin^2 t = 1$. Thus, the curves in Exploration 2 were either ellipses or portions of ellipses, each with center at the origin.

In the exercises you will see how to graph hyperbolas parametrically.

Lines and Other Curves

Lines, line segments, and many other curves can be defined parametrically.

$x = 3t,\ y = 2 - 2t$



[−4, 4] by [−2, 4]

Figure 1.28 The graph of the line segment $x = 3t$, $y = 2 - 2t$, $0 \le t \le 1$, with trace on the initial point $(0, 2)$. (Example 4)

EXAMPLE 4 Graphing a Line Segment

Draw and identify the graph of the parametric curve determined by

$$x = 3t, \quad y = 2 - 2t, \quad 0 \le t \le 1.$$

SOLUTION

The graph (Figure 1.28) appears to be a line segment with endpoints $(0, 2)$ and $(3, 0)$. When $t = 0$, the equations give $x = 0$ and $y = 2$. When $t = 1$, they give $x = 3$ and $y = 0$. When we substitute $t = x/3$ into the y equation, we obtain

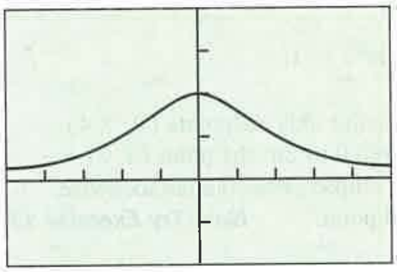
$$y = 2 - 2\left(\frac{x}{3}\right) = -\frac{2}{3}x + 2.$$

Thus, the parametric curve traces the segment of the line $y = -(2/3)x + 2$ from the point $(0, 2)$ to $(3, 0)$. ***Now Try Exercise 17.***

If we change the parameter interval $[0, 1]$ in Example 4 to $(-\infty, \infty)$, the parametrization will trace the entire line $y = -(2/3)x + 2$.

The bell-shaped curve in Exploration 3 is the famous witch of Agnesi. You will find more information about this curve in Exercise 47.

$x = 2 \cot t,\ y = 2 \sin^2 t$



[−5, 5] by [−2, 4]

Figure 1.29 The witch of Agnesi. (Exploration 3)

EXPLORATION 3 Graphing the Witch of Agnesi

The witch of Agnesi is the curve

$$x = 2 \cot t, \quad y = 2 \sin^2 t, \quad 0 < t < \pi.$$

1. Draw the curve using the window in Figure 1.29. What did you choose as a closed parameter interval for your grapher? In what direction is the curve traced? How far to the left and right of the origin do you think the curve extends?
2. Graph the same parametric equations using the parameter intervals $(-\pi/2, \pi/2)$, $(0, \pi/2)$, and $(\pi/2, \pi)$. In each case, describe the curve you see and the direction in which it is traced by your grapher.
3. What happens if you replace $x = 2 \cot t$ by $x = -2 \cot t$ in the original parametrization? What happens if you use $x = 2 \cot(\pi - t)$?

Maria Agnesi (1718–1799)

Analytical Institutions, the most complete treatment of calculus of its time, was written in Italian by Maria Agnesi and quickly translated into many other languages. Agnesi started writing it when she was 20, the same year she published her *Philosophical Propositions*, a series of essays on natural science and philosophy. At the age of 32 she was offered the position of Professor of Mathematics at the University of Bologna, the first woman to be offered such a position in Europe. She declined it, but remained an honorary faculty member.

Today, Agnesi is remembered chiefly for a bell-shaped curve called *the witch of Agnesi*. This name, found only in English texts, is the result of a mistranslation. Agnesi's own name for the curve was *versiera* or "turning curve." John Colson, a noted Cambridge mathematician, probably confused *versiera* with *avversiera*, which means "wife of the devil" and translated it into "witch."

EXAMPLE 5 Parametrizing a Line Segment

Find a parametrization for the line segment with endpoints $(-2, 1)$ and $(3, 5)$.

SOLUTION

Using $(-2, 1)$ we create the parametric equations

$$x = -2 + at, \qquad y = 1 + bt.$$

These represent a line, as we can see by solving each equation for t and equating to obtain

$$\frac{x + 2}{a} = \frac{y - 1}{b}.$$

continued

This line goes through the point $(-2, 1)$ when $t = 0$. We determine a and b so that the line goes through $(3, 5)$ when $t = 1$.

$$3 = -2 + a \quad \Rightarrow \quad a = 5 \qquad x = 3 \text{ when } t = 1.$$

$$5 = 1 + b \quad \Rightarrow \quad b = 4 \qquad y = 5 \text{ when } t = 1.$$

Therefore,

$$x = -2 + 5t, \quad y = 1 + 4t, \quad 0 \le t \le 1$$

is a parametrization of the line segment with initial point $(-2, 1)$ and terminal point $(3, 5)$.

Now Try Exercise 23.

Quick Review 1.4 *(For help, go to Section 1.1 and Appendix A1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–3, write an equation for the line.

1. the line through the points $(1, 8)$ and $(4, 3)$

2. the horizontal line through the point $(3, -4)$

3. the vertical line through the point $(2, -3)$

In Exercises 4–6, find the x- and y-intercepts of the graph of the relation.

4. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ **5.** $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

6. $2y^2 = x + 1$

In Exercises 7 and 8, determine whether the given points lie on the graph of the relation.

7. $2x^2y + y^2 = 3$

(a) $(1, 1)$ **(b)** $(-1, -1)$ **(c)** $(1/2, -2)$

8. $9x^2 - 18x + 4y^2 = 27$

(a) $(1, 3)$ **(b)** $(1, -3)$ **(c)** $(-1, 3)$

9. Solve for t.

(a) $2x + 3t = -5$ **(b)** $3y - 2t = -1$

10. For what values of a is each equation true?

(a) $\sqrt{a^2} = a$ **(b)** $\sqrt{a^2} = \pm a$

(c) $\sqrt{4a^2} = 2|a|$

Section 1.4 Exercises

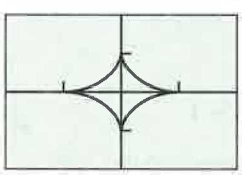
In Exercises 1–4, match the parametric equations with their graph. State the approximate dimensions of the viewing window. Give a parameter interval that traces the curve exactly once.

1. $x = 3 \sin(2t), \quad y = 1.5 \cos t$

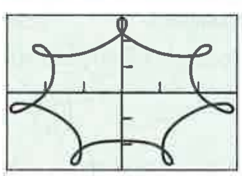
2. $x = \sin^3 t, \quad y = \cos^3 t$

3. $x = 7 \sin t - \sin(7t), \quad y = 7 \cos t - \cos(7t)$

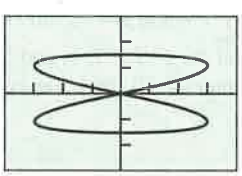
4. $x = 12 \sin t - 3 \sin(6t), \quad y = 12 \cos t + 3 \cos(6t)$



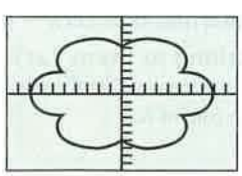
(a)



(b)



(c)



(d)

In Exercises 5–22, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

5. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

6. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

7. $x = t, \quad y = \sqrt{t}, \quad t \ge 0$

8. $x = (\sec^2 t) - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

9. $x = \cos t, \quad y = \sin t, \quad 0 \le t \le \pi$

10. $x = \sin(2\pi t), \quad y = \cos(2\pi t), \quad 0 \le t \le 1$

11. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

12. $x = 4 \cos t, \quad y = 2 \sin t, \quad 0 \le t \le 2\pi$

13. $x = 4 \sin t, \quad y = 2 \cos t, \quad 0 \le t \le \pi$

14. $x = 4 \sin t, \quad y = 5 \cos t, \quad 0 \le t \le 2\pi$

15. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$

16. $x = 1 - t, \quad y = 1 + t, \quad -\infty < t < \infty$

17. $x = t, \quad y = 1 - t, \quad 0 \leq t \leq 1$

18. $x = 3 - 3t, \quad y = 2t, \quad 0 \leq t \leq 1$

19. $x = 4 - \sqrt{t}, \quad y = \sqrt{t}, \quad 0 \leq t$

20. $x = t^2, \quad y = \sqrt{4 - t^2}, \quad 0 \leq t \leq 2$

21. $x = \sin t, \quad y = \cos 2t, \quad -\infty < t < \infty$

22. $x = t^2 - 3, \quad y = t, \quad t \leq 0$

In Exercises 23–28, find a parametrization for the curve.

23. the line segment with endpoints $(-1, -3)$ and $(4, 1)$

24. the line segment with endpoints $(-1, 3)$ and $(3, -2)$

25. the lower half of the parabola $x - 1 = y^2$

26. the left half of the parabola $y = x^2 + 2x$

27. the ray (half line) with initial point $(2, 3)$ that passes through the point $(-1, -1)$

28. the ray (half line) with initial point $(-1, 2)$ that passes through the point $(0, 0)$

Group Activity In Exercises 29–32, refer to the graph of

$$x = 3 - |t|, \quad y = t - 1, \quad -5 \leq t \leq 5,$$

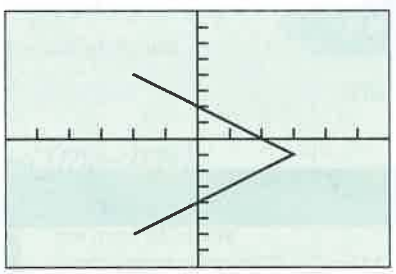
shown in the figure. Find the values of t that produce the graph in the given quadrant.

29. Quadrant I

30. Quadrant II

31. Quadrant III

32. Quadrant IV



[-6, 6] by [-8, 8]

In Exercises 33 and 34, find a parametrization for the part of the graph that lies in Quadrant I.

33. $y = x^2 + 2x + 2$

34. $y = \sqrt{x + 3}$

35. ***Circles*** Find parametrizations to model the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2, a > 0$, as indicated.

(a) once clockwise
(b) once counterclockwise
(c) twice clockwise
(d) twice counterclockwise

36. ***Ellipses*** Find parametrizations to model the motion of a particle that starts at $(-a, 0)$ and traces the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \quad a > 0, b > 0,$$

as indicated.

(a) once clockwise
(b) once counterclockwise
(c) twice clockwise
(d) twice counterclockwise

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

37. True or False The graph of the parametric curve $x = 3 \cos t$, $y = 4 \sin t$ is a circle. Justify your answer.

38. True or False The parametric curve $x = 2 \cos(-t)$, $y = 2 \sin(-t)$, $0 \leq t \leq 2\pi$ is traced clockwise. Justify your answer.

In Exercises 39 and 40, use the parametric curve $x = 5t$, $y = 3 - 3t$, $0 \leq t \leq 1$.

39. Multiple Choice Which of the following describes its graph?

(A) circle (B) parabola (C) ellipse
(D) line segment (E) line

40. Multiple Choice Which of the following is the initial point of the curve?

(A) $(-5, 6)$ (B) $(0, -3)$ (C) $(0, 3)$ (D) $(5, 0)$
(E) $(10, -3)$

41. Multiple Choice Which of the following describes the graph of the parametric curve $x = -3 \sin t$, $y = -3 \cos t$?

(A) circle (B) parabola (C) ellipse
(D) hyperbola (E) line

42. Multiple Choice Which of the following describes the graph of the parametric curve $x = 3t$, $y = 2t$, $t \geq 1$?

(A) circle (B) parabola (C) line segment
(D) line (E) ray

Explorations

43. ***Hyperbolas*** Let $x = a \sec t$ and $y = b \tan t$.

(a) **Writing to Learn** Let $a = 1, 2$, or 3, $b = 1, 2$, or 3, and graph using the parameter interval $(-\pi/2, \pi/2)$. Explain what you see, and describe the role of a and b in these parametric equations. [*Hint:* Recall that $\sec t = 1/\cos t$.]

(b) Let $a = 2$, $b = 3$, and graph in the parameter interval $(\pi/2, 3\pi/2)$. Explain what you see.

(c) **Writing to Learn** Let $a = 2$, $b = 3$, and graph using the parameter interval $(-\pi/2, 3\pi/2)$. The two lines that look like asymptotes are not really part of the graph; they show up because the grapher is in CONNECTED mode while drawing a disconnected curve. Give a more complete explanation for the appearance of these lines based on the parameter t.

(d) Use algebra to explain why

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

(e) Let $x = a \tan t$ and $y = b \sec t$. Repeat (a), (b), and (d) using an appropriate version of (d).

44. ***Transformations*** Let $x = (2 \cos t) + h$ and $y = (2 \sin t) + k$.

(a) **Writing to Learn** Let $k = 0$ and $h = -2, -1, 1$, and 2, in turn. Graph using the parameter interval $[0, 2\pi]$. Describe the role of h.

(b) **Writing to Learn** Let $h = 0$ and $k = -2, -1, 1,$ and 2, in turn. Graph using the parameter interval $[0, 2\pi]$. Describe the role of k.

(c) Find a parametrization for the circle with radius 5 and center at $(2, -3)$.

(d) Find a parametrization for the ellipse centered at $(-3, 4)$ with semimajor axis of length 5 parallel to the x-axis and semiminor axis of length 2 parallel to the y-axis.

In Exercises 45 and 46, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

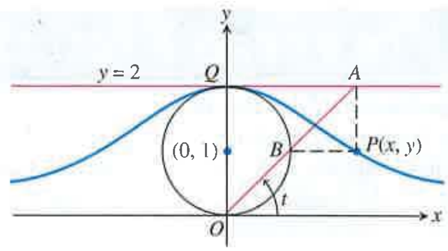
(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

45. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

46. $x = \tan t, \quad y = -2 \sec t, \quad -\pi/2 < t < \pi/2$

Extending the Ideas

47. *The Witch of Agnesi* The bell-shaped witch of Agnesi can be constructed as follows. Start with the circle of radius 1, centered at the point $(0, 1)$ as shown in the figure.



Choose a point A on the line $y = 2$, and connect it to the origin with a line segment. Call the point where the segment crosses the circle B. Let P be the point where the vertical line through A crosses the horizontal line through B. The witch is the curve traced by P as A moves along the line $y = 2$.

Find a parametrization for the witch by expressing the coordinates of P in terms of t, the radian measure of the angle that segment OA makes with the positive x-axis. The following equalities (which you may assume) will help:

(i) $x = AQ$ (ii) $y = 2 - AB \sin t$ (iii) $AB \cdot AO = (AQ)^2$

48. *Parametrizing Lines and Segments*

(a) Show that $x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t,$ $-\infty < t < \infty$ is a parametrization for the line through the points (x_1, y_1) and (x_2, y_2).

(b) Find a parametrization for the line segment with endpoints (x_1, y_1) and (x_2, y_2).