

## 1.3 Exponential Functions

**You will be able to** model exponential growth and decay with functions of the form  $y = k \cdot a^x$  and recognize exponential growth and decay in algebraic, numerical, and graphical representations.

- Exponential functions
- Rules of exponents
- Applications of exponential functions (growth and decay)
- Compound interest
- The number  $e$

### Exponential Growth

Table 1.1 shows the growth of \$100 invested in 1996 at an interest rate of 5.5%, compounded annually.

**TABLE 1.1 Savings Account Growth**

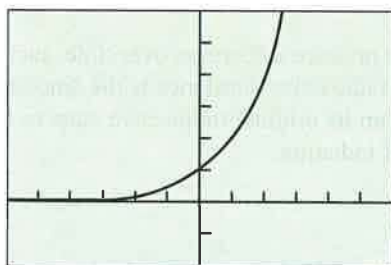
Year	Amount (dollars)	Increase (dollars)
1996	100	
1997	$100(1.055) = 105.50$	5.50
1998	$100(1.055)^2 = 111.30$	5.80
1999	$100(1.055)^3 = 117.42$	6.12
2000	$100(1.055)^4 = 123.88$	6.46

After the first year, the value of the account is always 1.055 times its value in the previous year. After  $n$  years, the value is  $y = 100 \cdot (1.055)^n$ .

Compound interest provides an example of *exponential growth* and is modeled by a function of the form  $y = P \cdot a^x$ , where  $P$  is the initial investment (called the *principal*) and  $a$  is equal to 1 plus the interest rate expressed as a decimal.

The equation  $y = P \cdot a^x$ ,  $a > 0$ ,  $a \neq 1$ , identifies a family of functions called *exponential functions*. Notice that the ratio of consecutive amounts in Table 1.1 is always the same:  $111.30/105.30 = 117.42/111.30 = 123.88/117.42 \approx 1.055$ . This fact is an important feature of exponential curves that has widespread application, as we will see.

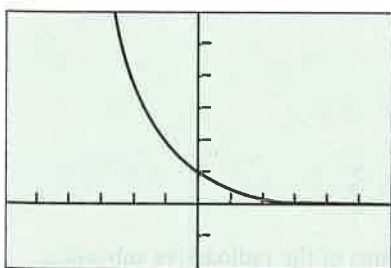
$$y = 2^x$$



$[-6, 6]$  by  $[-2, 6]$

(a)

$$y = 2^{-x}$$



$[-6, 6]$  by  $[-2, 6]$

(b)

**Figure 1.20** A graph of (a)  $y = 2^x$  and (b)  $y = 2^{-x}$ .

### EXPLORATION 1 Exponential Functions

1. Graph the function  $y = a^x$  for  $a = 2, 3, 5$ , in a  $[-5, 5]$  by  $[-2, 5]$  viewing window.
2. For what values of  $x$  is it true that  $2^x < 3^x < 5^x$ ?
3. For what values of  $x$  is it true that  $2^x > 3^x > 5^x$ ?
4. For what values of  $x$  is it true that  $2^x = 3^x = 5^x$ ?
5. Graph the function  $y = (1/a)^x = a^{-x}$  for  $a = 2, 3, 5$ .
6. Repeat parts 2–4 for the functions in part 5.

### DEFINITION Exponential Function

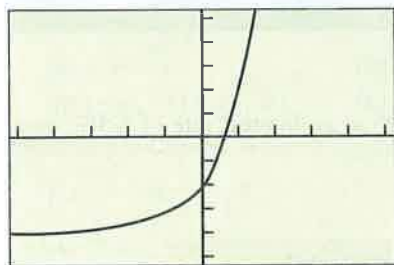
Let  $a$  be a positive real number other than 1. The function

$$f(x) = a^x$$

is the **exponential function with base  $a$** .

The domain of  $f(x) = a^x$  is  $(-\infty, \infty)$  and the range is  $(0, \infty)$ . If  $a > 1$ , the graph of  $f$  looks like the graph of  $y = 2^x$  in Figure 1.20a. If  $0 < a < 1$ , the graph of  $f$  looks like the graph of  $y = 2^{-x}$  in Figure 1.20b.

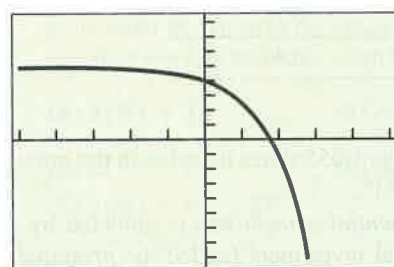
$$y = 2(3^x) - 4$$



[-5, 5] by [-5, 5]

**Figure 1.21** The graph of  $y = 2(3^x) - 4$ . (Example 1)

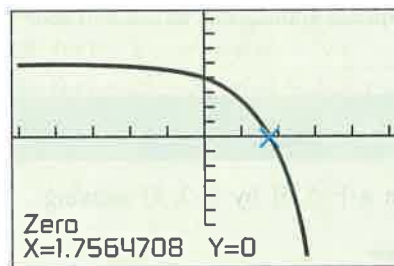
$$y = 5 - 2.5^x$$



[-5, 5] by [-8, 8]

(a)

$$y = 5 - 2.5^x$$

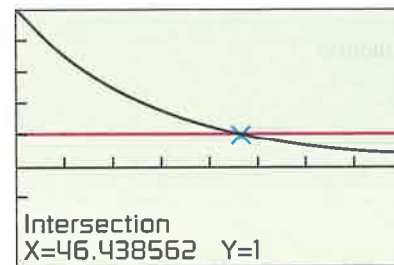


[-5, 5] by [-8, 8]

(b)

**Figure 1.22** (a) A graph of  $f(x) = 5 - 2.5^x$ . (b) Showing the use of the ZERO feature to approximate the zero of  $f$ . (Example 2)

$$y = 5\left(\frac{1}{2}\right)^{t/20}, y = 1$$



[0, 80] by [-3, 5]

**Figure 1.23** (Example 3)

### EXAMPLE 1 Graphing an Exponential Function

Graph the function  $y = 2(3^x) - 4$ . State its domain and range.

#### SOLUTION

Figure 1.21 shows the graph of the function  $y$ . It appears that the domain is  $(-\infty, \infty)$ . The range is  $(-4, \infty)$  because  $2(3^x) > 0$  for all  $x$ .

**Now Try Exercise 1.**

### EXAMPLE 2 Finding Zeros

Find the zeros of  $f(x) = 5 - 2.5^x$  graphically.

#### SOLUTION

Figure 1.22a suggests that  $f$  has a zero between  $x = 1$  and  $x = 2$ , closer to 2. We can use our grapher to find that the zero is approximately 1.756 (Figure 1.22b).

**Now Try Exercise 9.**

Exponential functions obey the rules for exponents.

#### Rules for Exponents

If  $a > 0$  and  $b > 0$ , the following hold for all real numbers  $x$  and  $y$ .

1.  $a^x \cdot a^y = a^{x+y}$
2.  $\frac{a^x}{a^y} = a^{x-y}$
3.  $(a^x)^y = (a^y)^x = a^{xy}$
4.  $a^x \cdot b^x = (ab)^x$
5.  $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

## Exponential Decay

Exponential functions can also model phenomena that produce a decrease over time, such as happens with radioactive decay. The **half-life** of a radioactive substance is the amount of time it takes for half of the substance to change from its original radioactive state to a nonradioactive state by emitting energy in the form of radiation.

### EXAMPLE 3 Modeling Radioactive Decay

Suppose the half-life of a certain radioactive substance is 20 days and that there are 5 grams present initially. When will there be only 1 gram of the substance remaining?

#### SOLUTION

The number of grams remaining after 20 days is

$$5\left(\frac{1}{2}\right) = \frac{5}{2}.$$

The number of grams remaining after 40 days is

$$5\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4}.$$

The function  $y = 5(1/2)^{t/20}$  models the mass in grams of the radioactive substance after  $t$  days. Figure 1.23 shows that the graphs of  $y_1 = 5(1/2)^{t/20}$  and  $y_2 = 1$  (for 1 gram) intersect when  $t$  is approximately 46.44.

There will be 1 gram of the radioactive substance left after approximately 46.44 days, or about 46 days 10.5 hours.

**Now Try Exercise 23.**

Compound interest investments, population growth, and radioactive decay are all examples of *exponential growth and decay*.

### DEFINITIONS Exponential Growth, Exponential Decay

The function  $y = k \cdot a^x$ ,  $k > 0$  is a model for **exponential growth** if  $a > 1$ , and a model for **exponential decay** if  $0 < a < 1$ .

## Compound Interest

One common application of exponential growth in the financial world is the compounding effect of accrued interest in a savings account. We saw at the beginning of this section that an account paying 5.5% annual interest would multiply by a factor of 1.055 every year, so an account starting with principal  $P$  would be worth  $P(1.055)^t$  after  $t$  years.

Some accounts pay interest multiple times per year, which increases the compounding effect. An account earning 6% annual interest compounded monthly would pay one-twelfth of the interest each month, effectively a monthly interest rate of 0.5%. Thus, after  $t$  years, the account would be worth  $P(1.005)^{12t}$ . The general formula is given below.

### Compound Interest Formula

If an account begins with principal  $P$  and earns an annual interest rate  $r$  compounded  $n$  times per year, then the value of the account in  $t$  years is

$$P \left( 1 + \frac{r}{n} \right)^{nt}$$

### EXAMPLE 4 Compound Interest

Bernie deposits \$2500 in an account earning 5.4% annual interest. Find the value of the account in 10 years if the interest is compounded

- (a) annually;      (b) quarterly;      (c) monthly.

#### SOLUTION

(a)  $2500(1 + 0.054)^{10} \approx 4230.06$ , so the account is worth \$4230.06.

(b)  $2500 \left( 1 + \frac{0.054}{4} \right)^{4(10)} \approx 4274.55$ , so the account is worth \$4274.55.

(c)  $2500 \left( 1 + \frac{0.054}{12} \right)^{12(10)} \approx 4284.82$ , so the account is worth \$4284.82.

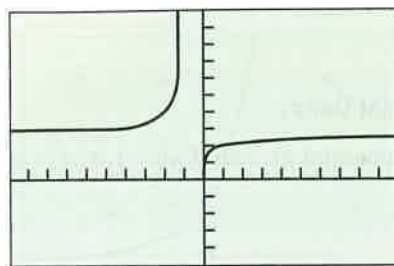
*Now Try Exercises 25 and 26.*

## The Number $e$

You may have noticed in Example 4 that the value of the account becomes greater as the number of interest payments per year increases, but the value does not appear to be increasing without bound. In fact, a bank can even offer to compound the interest *continuously* without paying that much more per year. This is because the number that serves as the base of the exponential function in the compound interest formula has a finite limit as  $n$  approaches infinity:

$$\left( 1 + \frac{r}{n} \right)^n \rightarrow e^r \quad \text{as } n \rightarrow \infty$$

$$y = (1 + 1/x)^x$$



$[-10, 10]$  by  $[-5, 10]$

X	Y <sub>1</sub>
1000	2.7169
2000	2.7176
3000	2.7178
4000	2.7179
5000	2.718
6000	2.7181
7000	2.7181

$$Y_1 = (1 + 1/X)^X$$

**Figure 1.24** A graph and table of values for  $f(x) = (1 + 1/x)^x$  both suggest that as  $x \rightarrow \infty$ ,  $f(x) \rightarrow e \approx 2.718$ .

You can get an estimate of the number  $e$  by looking at values of  $(1 + \frac{1}{x})^x$  for increasing values of  $x$ , as shown graphically and numerically in Figure 1.24. This number  $e$ , which is approximately 2.71828, is one of the most important numbers in mathematics. Like  $\pi$ , it is an irrational number that shows up in many different contexts, some of them quite surprising. You will encounter several of them in this course.

One place that  $e$  appears is in the formula for continuously compounded interest.

### Continuously Compounded Interest Formula

If an account begins with principal  $P$  and earns an annual interest rate  $r$  compounded continuously, then the value of the account in  $t$  years is

$$P(e^r)^t = Pe^{rt}.$$

### EXAMPLE 5 Continuously Compounded Interest

Bernice deposits \$2500 in an account earning 5.4% annual interest. Find the value of the account in 10 years if the interest is compounded

- (a) daily;      (b) continuously.

#### SOLUTION

(a)  $2500 \left(1 + \frac{0.054}{365}\right)^{365(10)} \approx 4289.85$ , so the account is worth \$4289.85.

(b)  $2500e^{0.054(10)} \approx 4290.02$ , so the account is worth \$4290.02.

Notice that the difference between “daily” and “continuously” for Bernice is 17 cents!

**Now Try Exercise 27.**

## Quick Review 1.3 (For help, go to Section 1.3.)

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–3, evaluate the expression. Round your answers to 3 decimal places.

1.  $5^{2/3}$

2.  $3^{\sqrt{2}}$

3.  $3^{-1.5}$

In Exercises 4–6, solve the equation. Round your answers to 4 decimal places.

4.  $x^3 = 17$

5.  $x^5 = 24$

6.  $x^{10} = 1.4567$

In Exercises 7 and 8, find the value of investing  $P$  dollars for  $n$  years with the interest rate  $r$  compounded annually.

7.  $P = \$500$ ,  $r = 4.75\%$ ,  $n = 5$  years

8.  $P = \$1000$ ,  $r = 6.3\%$ ,  $n = 3$  years

In Exercises 9 and 10, simplify the exponential expression.

9.  $\frac{(x^{-3}y^2)^2}{(x^4y^3)^3}$

10.  $\left(\frac{a^3b^{-2}}{c^4}\right)^2 \left(\frac{a^4c^{-2}}{b^3}\right)^{-1}$

## Section 1.3 Exercises

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, graph the function. State its domain and range.

1.  $y = -2^x + 3$

2.  $y = e^x + 3$

3.  $y = 3 \cdot e^{-x} - 2$

4.  $y = -2^{-x} - 1$

In Exercises 5–8, rewrite the exponential expression to have the indicated base.

5.  $9^{2x}$ , base 3

6.  $16^{3x}$ , base 2

7.  $(1/8)^{2x}$ , base 2

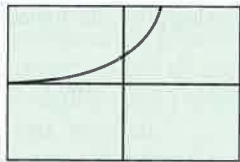
8.  $(1/27)^x$ , base 3

In Exercises 9–12, use a graph to find the zeros of the function.

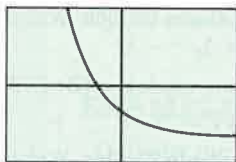
9.  $f(x) = 2^x - 5$                       10.  $f(x) = e^x - 4$   
 11.  $f(x) = 3^x - 0.5$                     12.  $f(x) = 3 - 2^x$

In Exercises 13–18, match the function with its graph. Try to do it without using your grapher.

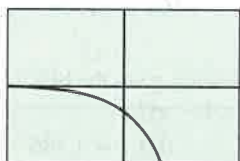
13.  $y = 2^x$                       14.  $y = 3^{-x}$                       15.  $y = -3^{-x}$   
 16.  $y = -0.5^{-x}$                 17.  $y = 2^{-x} - 2$                 18.  $y = 1.5^x - 2$



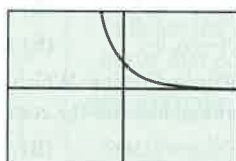
(a)



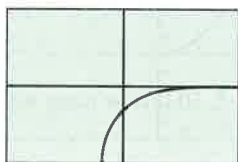
(b)



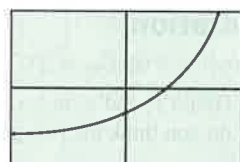
(c)



(d)



(e)



(f)

In Exercises 19–32, use an exponential model to solve the problem.

19. **Population Growth** The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?  
 20. **Population Growth** The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.  
 (a) Estimate the population in 1915 and 1940.  
 (b) Approximately when did the population reach 50,000?  
 21. **Half-life** The approximate half-life of titanium-44 is 63 years. How long will it take a sample to lose  
 (a) 50% of its titanium-44?  
 (b) 75% of its titanium-44?  
 22. **Half-life** The amount of silicon-32 in a sample will decay from 28 grams to 7 grams in approximately 340 years. What is the approximate half-life of silicon-32?  
 23. **Radioactive Decay** The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.  
 (a) Express the amount of phosphorus-32 remaining as a function of time  $t$ .  
 (b) When will there be 1 gram remaining?  
 24. **Finding Time** If John invests \$2300 in a savings account with a 6% interest rate compounded annually, how long will it take until John's account has a balance of \$4150?

25. **Doubling Your Money** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.  
 26. **Doubling Your Money** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded monthly.  
 27. **Doubling Your Money** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded continuously.  
 28. **Tripling Your Money** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded annually.  
 29. **Tripling Your Money** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded daily.  
 30. **Tripling Your Money** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.  
 31. **Cholera Bacteria** Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 h?  
 32. **Eliminating a Disease** Suppose that in any given year, the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take  
 (a) to reduce the number of cases to 1000?  
 (b) to eliminate the disease; that is, to reduce the number of cases to less than 1?

**Group Activity** In Exercises 33–36, copy and complete the table for the function.

33.  $y = 2x - 3$

$x$	$y$	Change ( $\Delta y$ )
1	?	?
2	?	?
3	?	?
4	?	?

34.  $y = -3x + 4$

$x$	$y$	Change ( $\Delta y$ )
1	?	?
2	?	?
3	?	?
4	?	?

35.  $y = x^2$

$x$	$y$	Change ( $\Delta y$ )
1	?	?
2	?	?
3	?	?
4	?	?

36.  $y = 3e^x$

$x$	$y$	Ratio ( $y_i/y_{i-1}$ )
1	?	?
2	?	?
3	?	?
4	?	?

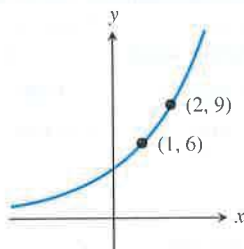
37. **Writing to Learn** Explain how the change  $\Delta y$  is related to the slopes of the lines in Exercises 33 and 34. If the changes in  $x$  are constant for a linear function, what would you conclude about the corresponding changes in  $y$ ?

38. **Bacteria Growth** The number of bacteria in a petri dish culture after  $t$  hours is

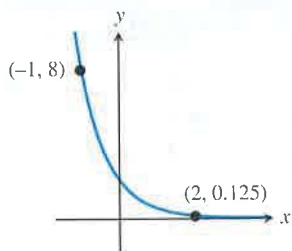
$$B = 100e^{0.693t}$$

- What was the initial number of bacteria present?
- How many bacteria are present after 6 hours?
- Approximately when will the number of bacteria be 200? Estimate the doubling time of the bacteria.

39. The graph of the exponential function  $y = a \cdot b^x$  is shown below. Find  $a$  and  $b$ .



40. The graph of the exponential function  $y = a \cdot b^x$  is shown below. Find  $a$  and  $b$ .



## Standardized Test Questions

You may use a graphing calculator to solve the following problems.

41. **True or False** The number  $3^{-2}$  is negative. Justify your answer.
42. **True or False** If  $4^3 = 2^a$ , then  $a = 6$ . Justify your answer.
43. **Multiple Choice** John invests \$200 at 4.5% compounded annually. About how long will it take for John's investment to double in value?  
 (A) 6 yr (B) 9 yr (C) 12 yr (D) 16 yr (E) 20 yr
44. **Multiple Choice** Which of the following gives the domain of  $y = 2e^{-x} - 3$ ?  
 (A)  $(-\infty, \infty)$  (B)  $[-3, \infty)$  (C)  $[-1, \infty)$  (D)  $(-\infty, 3]$   
 (E)  $x \neq 0$
45. **Multiple Choice** Which of the following gives the range of  $y = 4 - 2^{2x}$ ?  
 (A)  $(-\infty, \infty)$  (B)  $(-\infty, 4)$  (C)  $[-4, \infty)$   
 (D)  $(-\infty, 4]$  (E) all reals
46. **Multiple Choice** Which of the following gives the best approximation for the zero of  $f(x) = 4 - e^{3x}$ ?  
 (A)  $x = -1.386$  (B)  $x = 0.386$  (C)  $x = 1.386$   
 (D)  $x = 3$  (E) There are no zeros.

## Exploration

47. Let  $y_1 = x^2$  and  $y_2 = 2^x$ .
- Graph  $y_1$  and  $y_2$  in  $[-5, 5]$  by  $[-2, 10]$ . How many times do you think the two graphs cross?
  - Compare the corresponding changes in  $y_1$  and  $y_2$  as  $x$  changes from 1 to 2, 2 to 3, and so on. How large must  $x$  be for the changes in  $y_2$  to overtake the changes in  $y_1$ ?
  - Solve for  $x$ :  $x^2 = 2^x$ .
  - Solve for  $x$ :  $x^2 < 2^x$ .

## Extending the Ideas

In Exercises 48 and 49, assume that the graph of the exponential function  $f(x) = k \cdot a^x$  passes through the two points. Find the values of  $a$  and  $k$ .

48.  $(1, 4.5), (-1, 0.5)$
49.  $(1, 1.5), (-1, 6)$

## Quick Quiz for AP\* Preparation: Sections 1.1–1.3

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** Which of the following gives an equation for the line through  $(3, -1)$  and parallel to the line  $y = -2x + 1$ ?  
 (A)  $y = \frac{1}{2}x + \frac{7}{2}$  (B)  $y = \frac{1}{2}x - \frac{5}{2}$  (C)  $y = -2x + 5$   
 (D)  $y = -2x - 7$  (E)  $y = -2x + 1$
2. **Multiple Choice** If  $f(x) = x^2 + 1$  and  $g(x) = 2x - 1$ , which of the following gives  $(f \circ g)(2)$ ?  
 (A) 2 (B) 5 (C) 9 (D) 10 (E) 15
3. **Multiple Choice** The half-life of a certain radioactive substance is 8 hr. There are 5 grams present initially. Which of the following gives the best approximation when there will be 1 gram remaining?  
 (A) 2 (B) 10 (C) 15 (D) 16 (E) 19
4. **Free Response** Let  $f(x) = e^{-x} - 2$ .
- Find the domain of  $f$ .
  - Find the range of  $f$ .
  - Find the zeros of  $f$ .